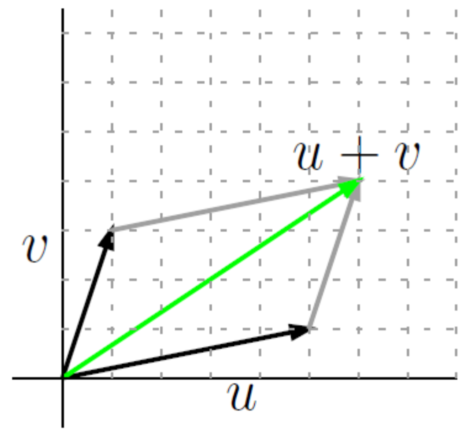
Elementary maths for GMT

Linear Algebra

Part 3: Transformations

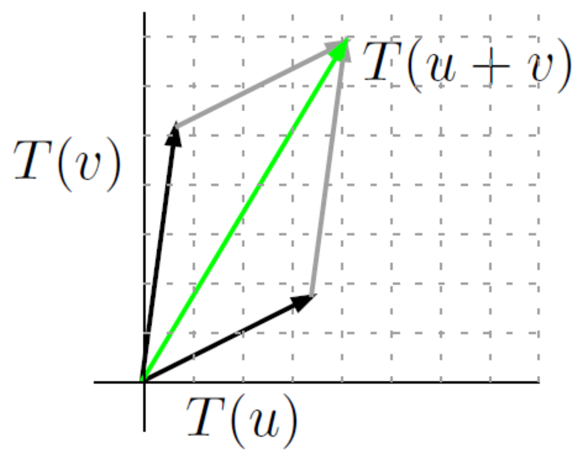
Linear transformations

- A function $T: \mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation if it satisfies:



  1. $T(u + v) = T(u) + T(v)$ for all $u, v \in \mathbb{R}^n$

  2. $T(cv) = cT(v)$ for all $v \in \mathbb{R}^n$ and all scalars c



Linear transformations in graphics

- Many transformations that we use in graphics are linear transformations
- Linear transformations can be represented by matrices
- A sequence of linear transformations can be represented with a single matrix
- With some tricks, we can represent translations and perspective projections with matrices as well

Matrices and linear transformations

- A 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ represents the linear transformation that maps the vector $(x \quad y)^T$ to the vector $(ax + by \quad cx + dy)^T$

Or (more readable): $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$

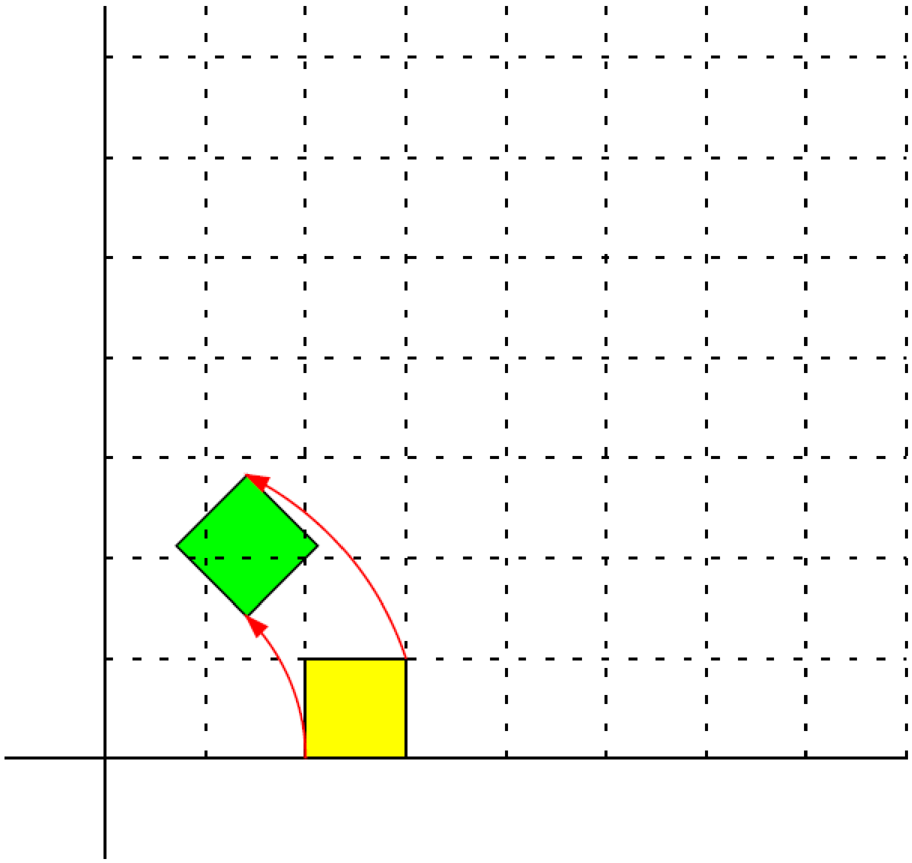
- A 2×3 matrix is a linear transformation that maps a 3D vector to a 2D vector (from some 3-dim. space to some 2-dim. plane)

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax + by + cz \\ dx + ey + fz \end{bmatrix}$$

Example: rotation

- To rotate $45°$ about the origin, we apply the matrix

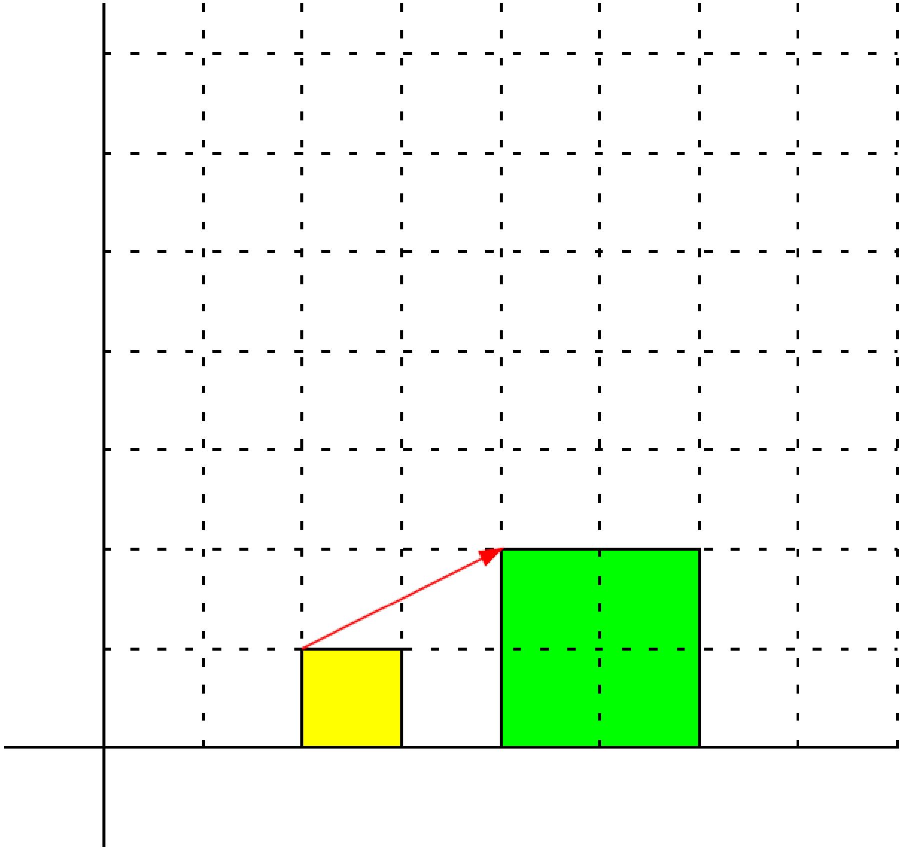
$$\begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$$



Example: scaling

- To scale with a factor two with respect to the origin, we apply the matrix

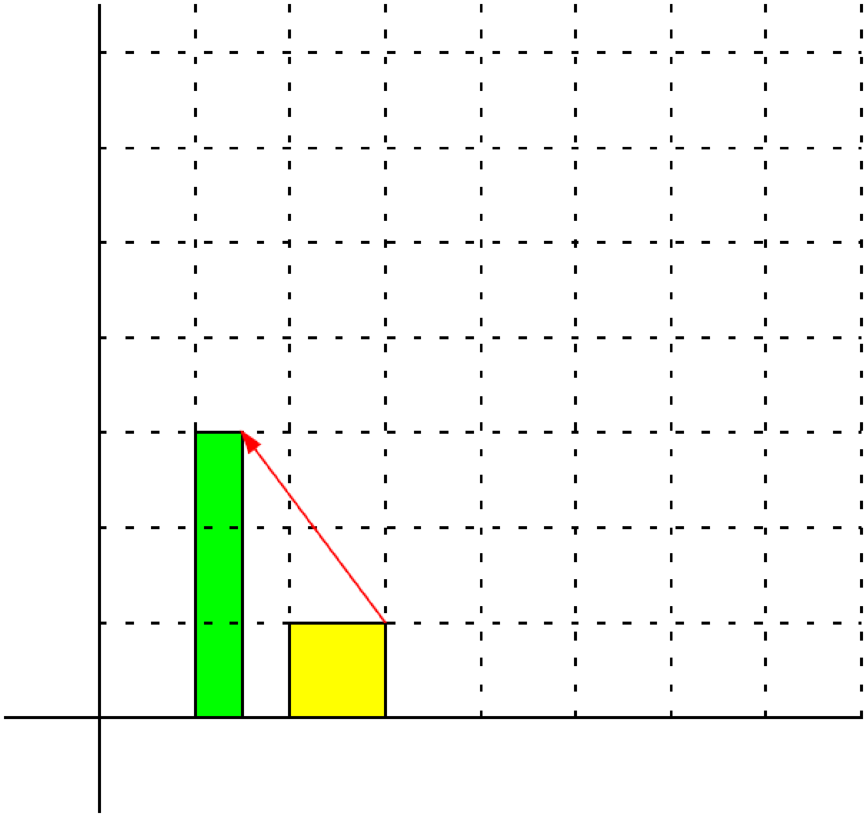
$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$



Example: scaling

- Scaling does not have to be uniform
- Here, we scale with a factor one half in x-direction and three in y-direction

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix}$$

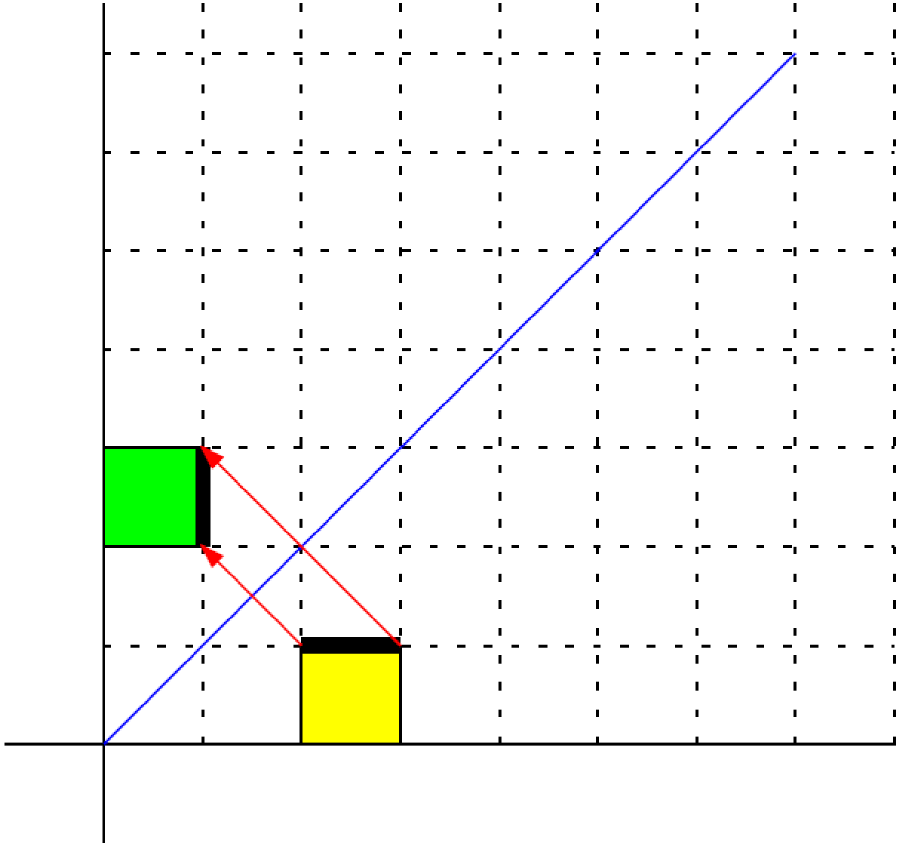


- Q: What is the inverse of this matrix?

Example: reflection

- Reflection in the line $y = x$ boils down to swapping x- and y- coordinates

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

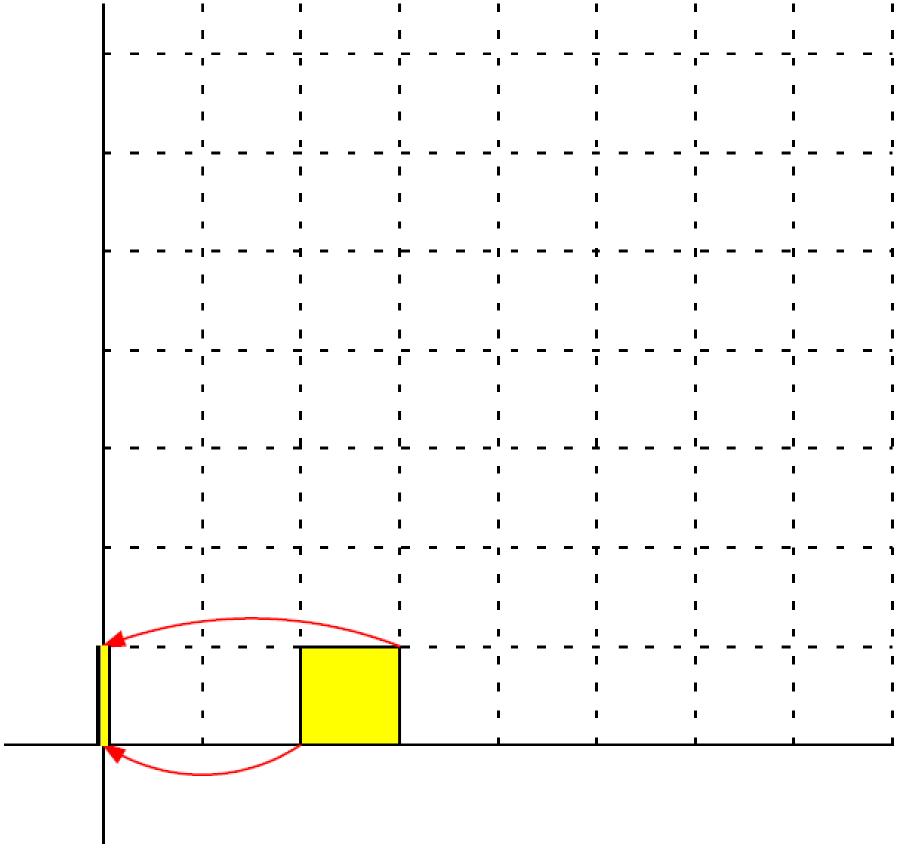


- Q: What is the inverse of this matrix?

Example: projection

- We can also use matrices to do orthographic projections, for instance onto the Y-axis

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

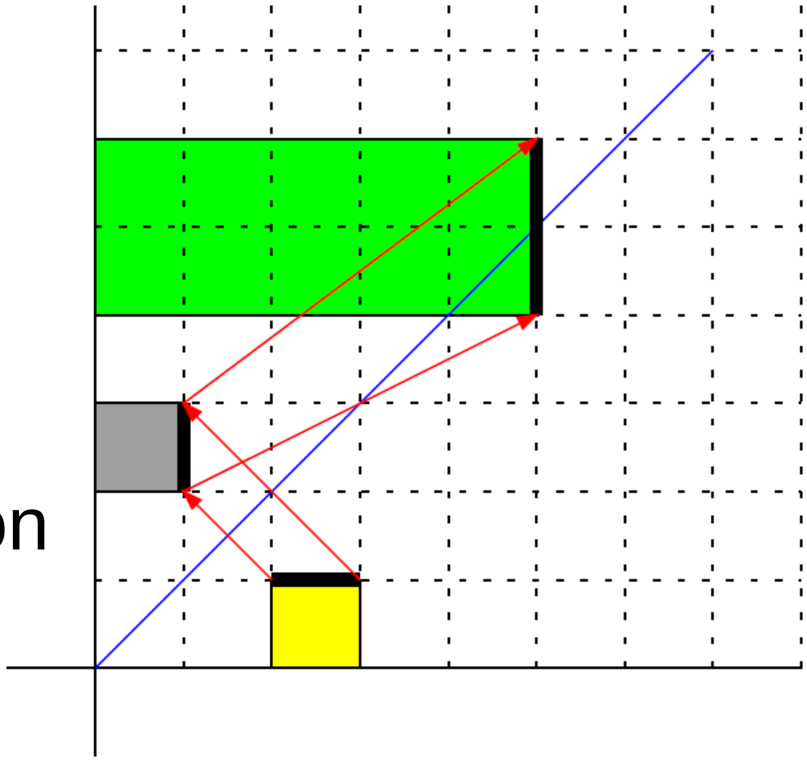


- Q: What is the inverse of the matrix?

Example: reflection and scaling

- Multiple transformations can be combined into one
- Here, we first do a reflection in the line $y = x$, and then we scale with a factor 5 in x-direction, and a factor 2 in y-direction

$$\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 2 & 0 \end{bmatrix}$$

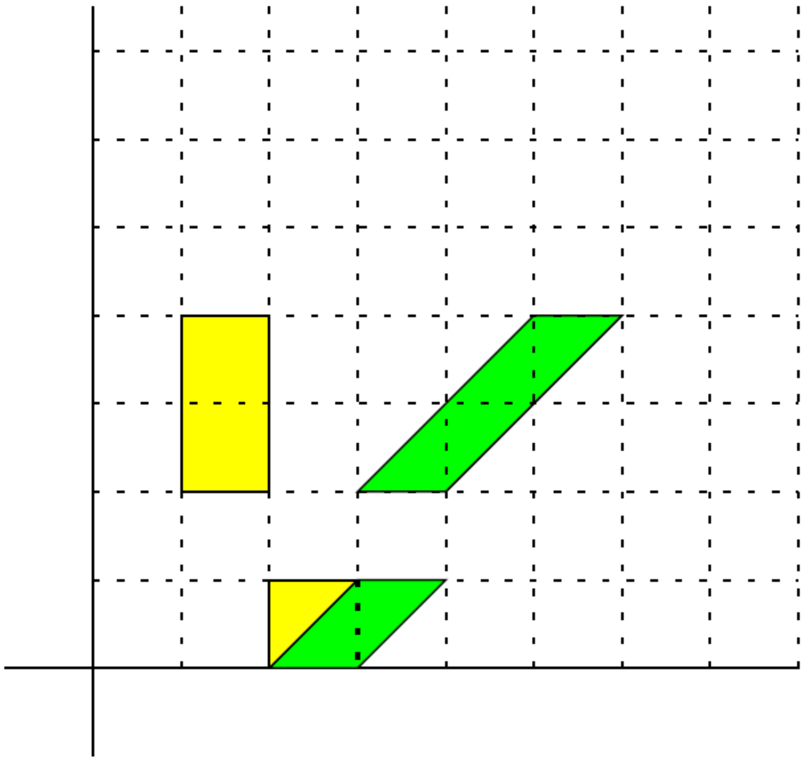


- Q: Why is the transformation that is done first rightmost?

Example: shearing

- Shearing in x-direction pushes things sideways

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

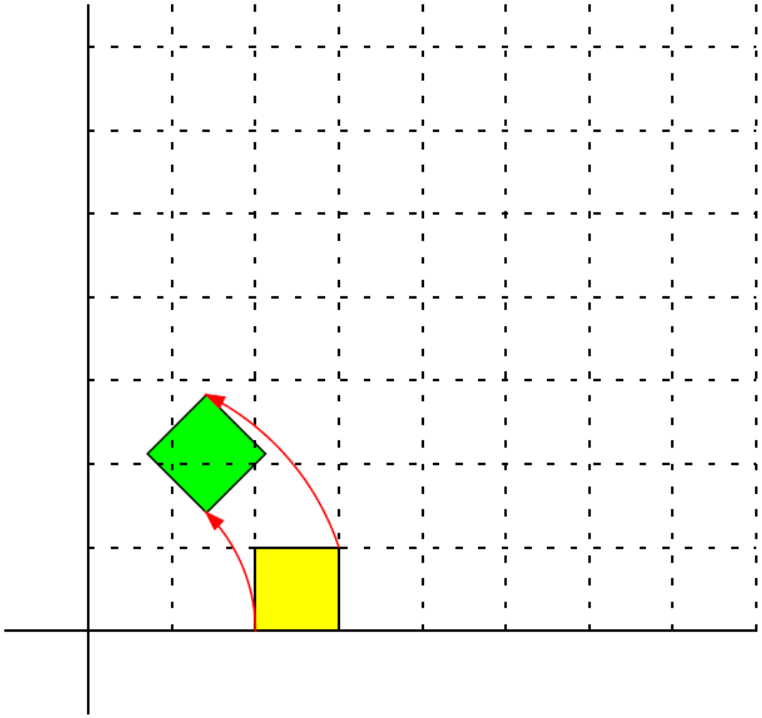


- Q: What happens with the x-coordinate of points that are transformed with this matrix? And what with the y-coordinates?
  What is the inverse of this matrix?

Finding matrices

- Applying matrices is pretty straightforward, but how de we find the matrix for a given linear transformation?

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$



- Q: What is the significance of the column vectors of A?

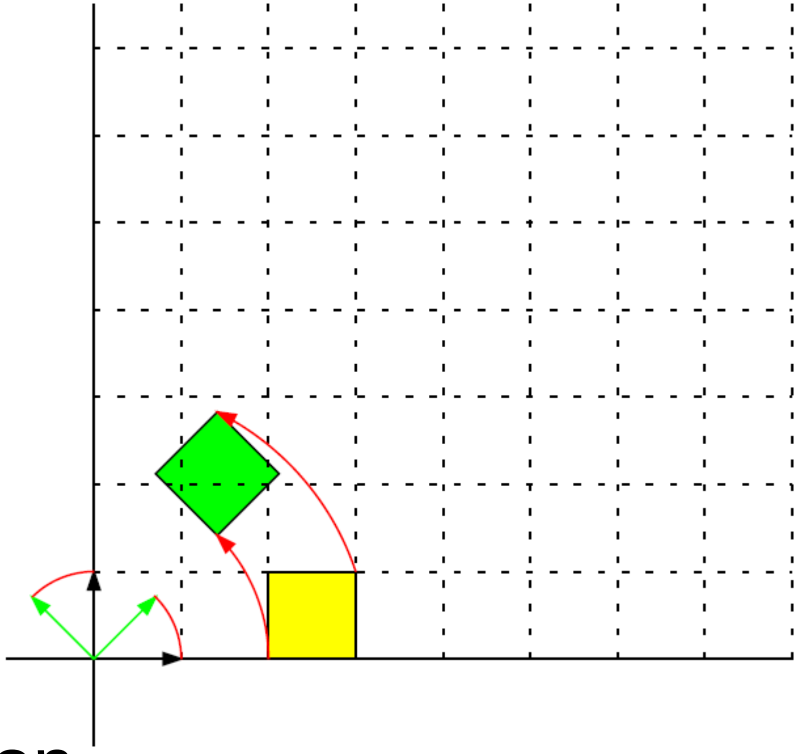
Finding matrices

- The column vectors of a transformation matrix are the images of the base vectors!

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \text{ and}$$

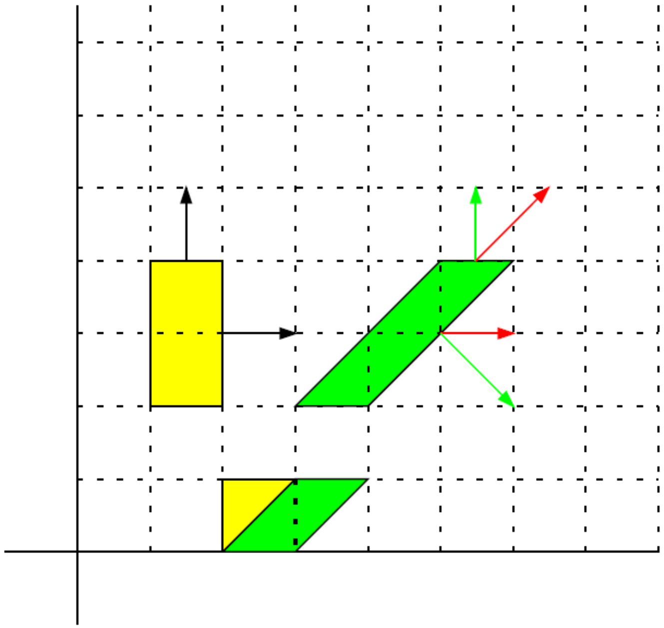
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

- That gives us an easy method of finding the matrix for a given linear transformation



Transposing normal vectors

- Unfortunately, normal vectors are not always transformed properly
- To transform a normal vector n under a given linear transformation A, we have to apply the matrix $(A^{-1})^T$



- Q: Obviously, for shearing, normal vectors 'behave funny'. But what about rotations? And scaling (uniform and non-uniform)?

Area and determinant

- For any linear transformation, the absolute value of the determinant represents the size change

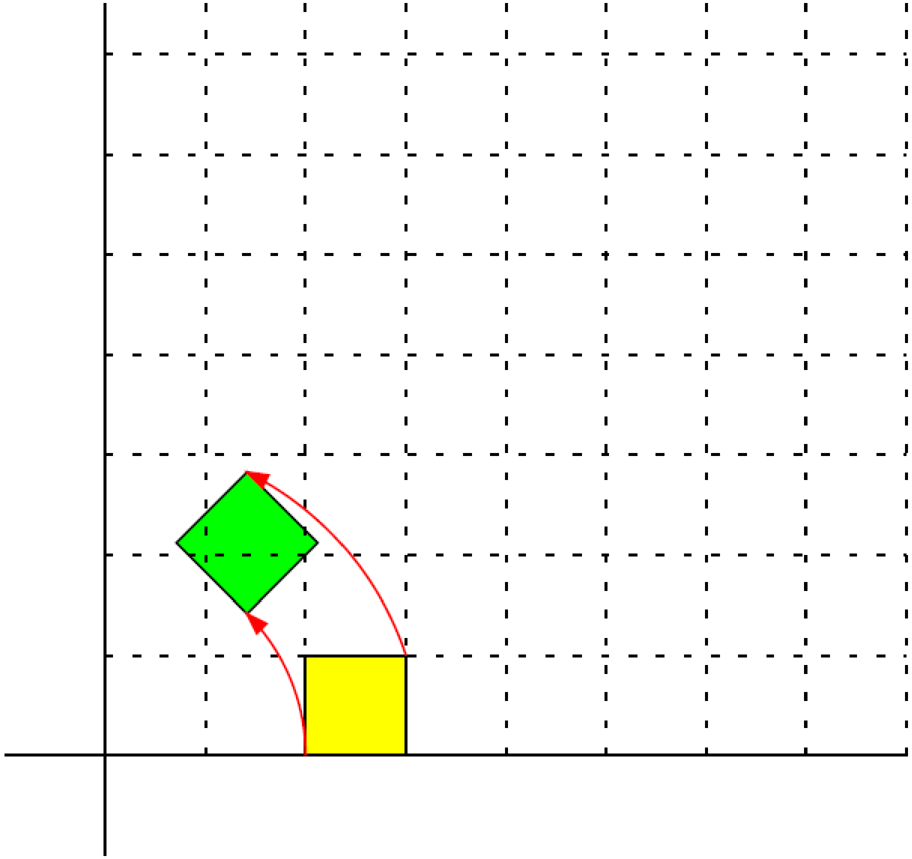
- For example, if a 2×2 matrix has determinant 3 or -3, then the linear transformation transforms a unit square to a shape with area 3

- Q: What is going on when the determinant is zero?

Example: rotation

- To rotate $45°$ about the origin, we apply the matrix

$$\begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$$



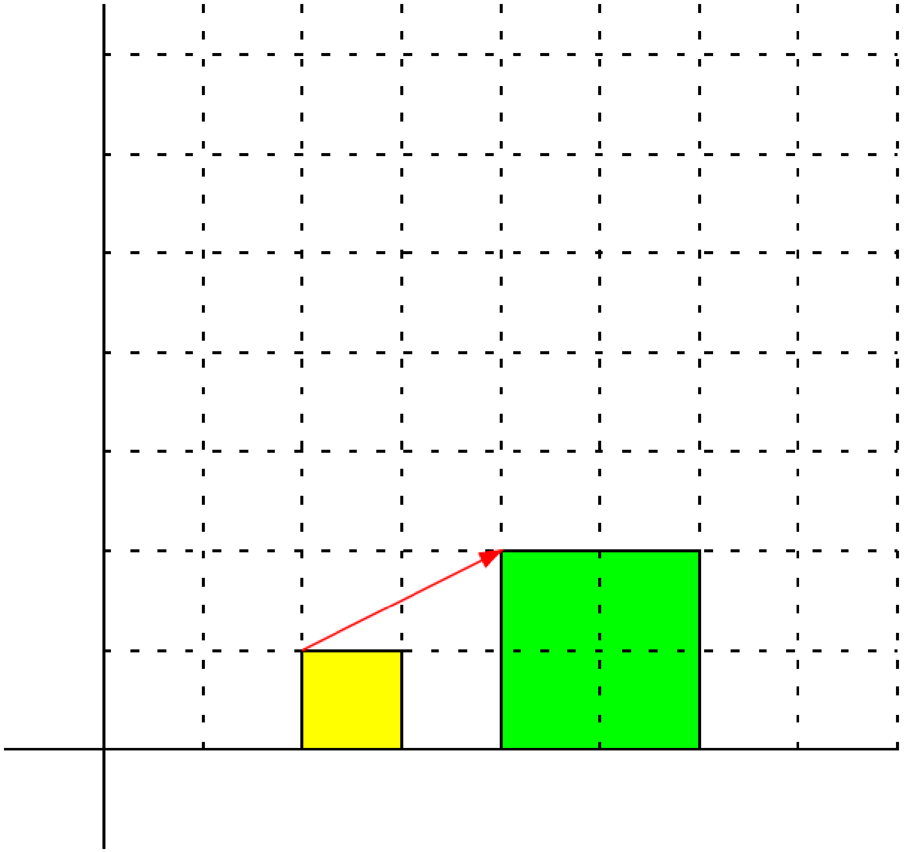
- Q: What is the determinant?

Example: scaling

- To scale with a factor two with respect to the origin, we apply the matrix

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

- Q: What is the determinant?



Example: scaling

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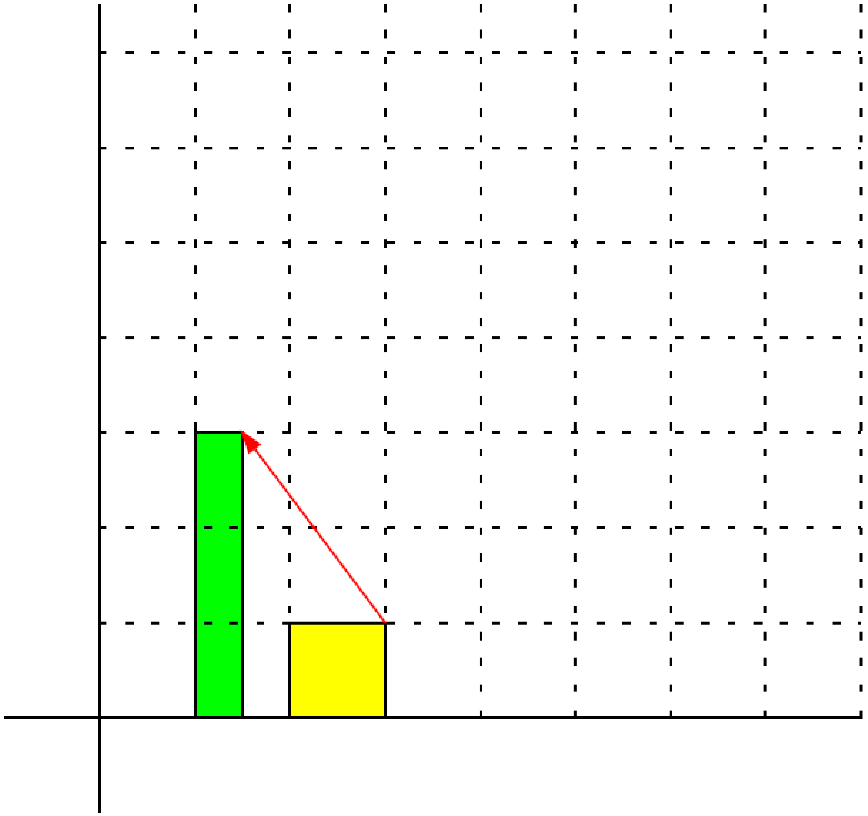
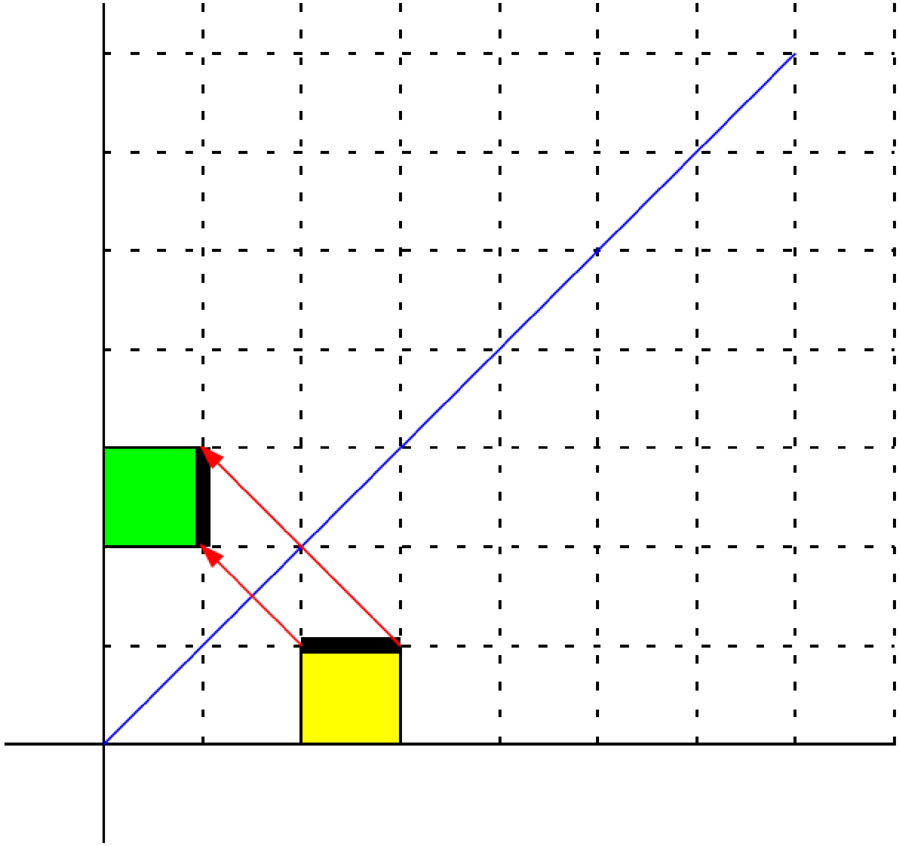


- Q: What is the determinant?

Example: reflection

- Reflection in the line $y = x$ boils down to swapping x- and y- coordinates

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

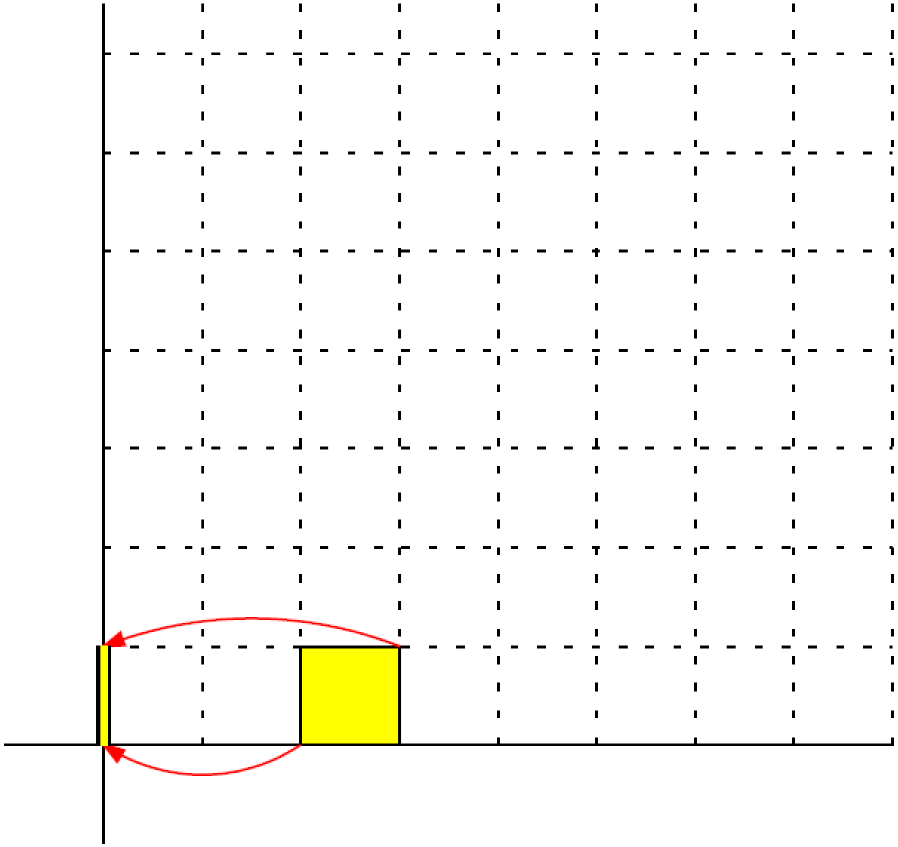


- Q: What is the determinant?

Example: projection

- We can also use matrices to do orthographic projections, for instance onto the Y-axis

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$



- Q: What is the determinant?

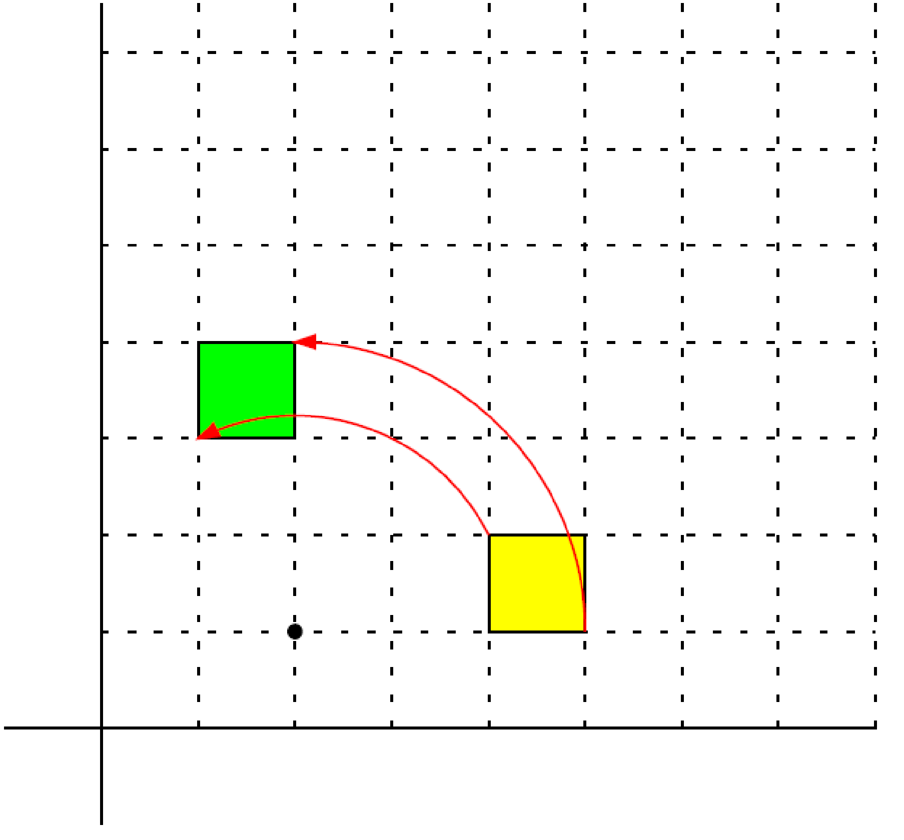
Determinant = 0

- The following statements are equivalent for a $n \times n$ matrix A and the linear transformation it represents:
  1. The determinant of A is zero
  2. The column vectors of A are linearly dependent
  3. The image space of the transformation is at most $(n-1)$-dimensional (the transformation is a projection)

More complex transformations

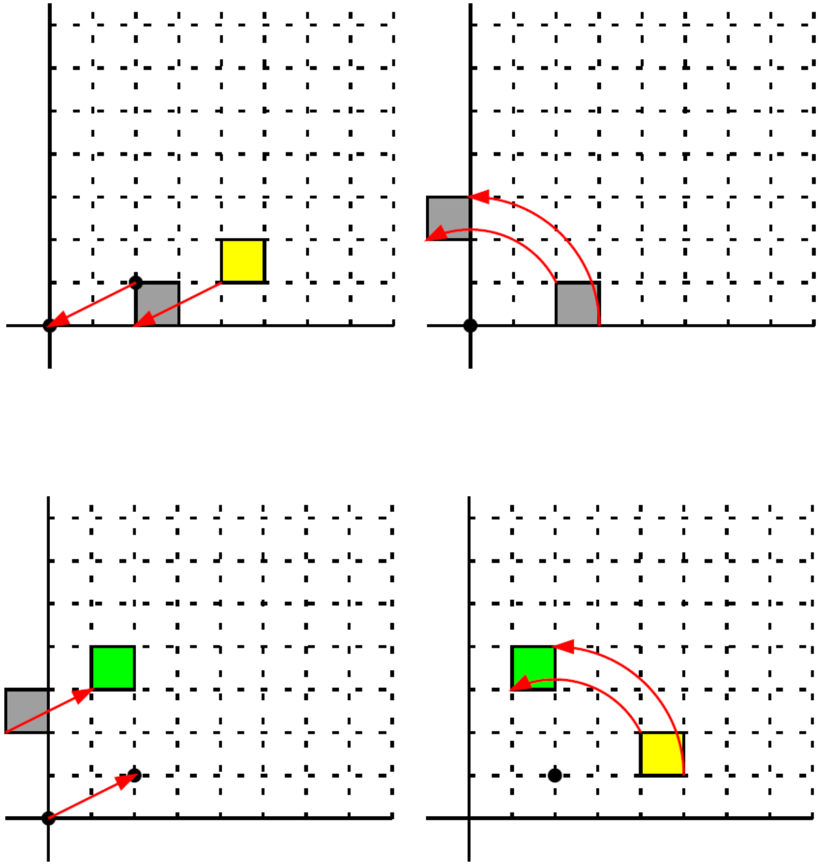
- So now we know how to determine matrices for a given transformation

- Let's try another one: what is the matrix for a rotation of $90°$ about the point $(2,1)$?



More complex transformations

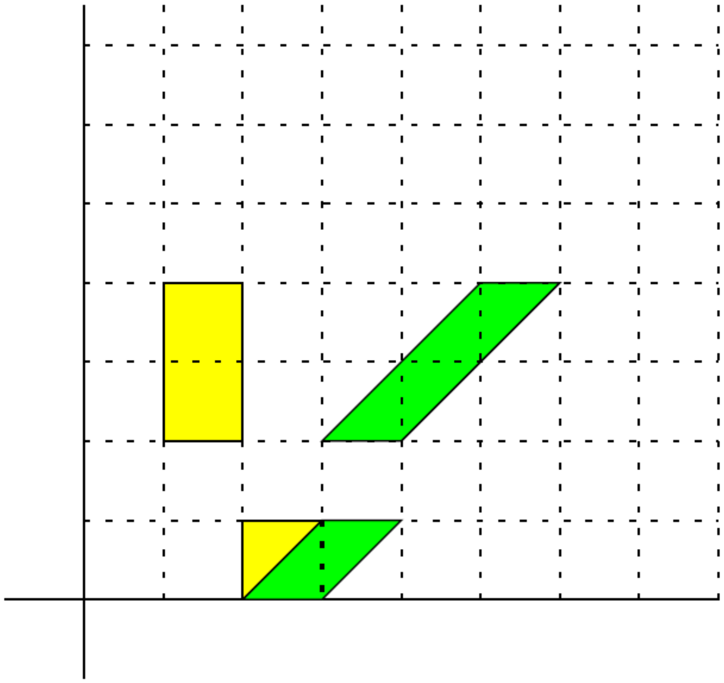
- We can build our transformation by composing three simpler transformations
  - Translate everything such that the center of rotation maps to the origin
  - Rotate about the origin
  - Revert the translation from the first step
- Q: But what is the matrix for a translation?



Homogeneous coordinates

- Translation is not a linear transformation
- A combination of linear transformations and translations is called an affine transformation

- But shearing in 2D looks a lot like translation in 1D



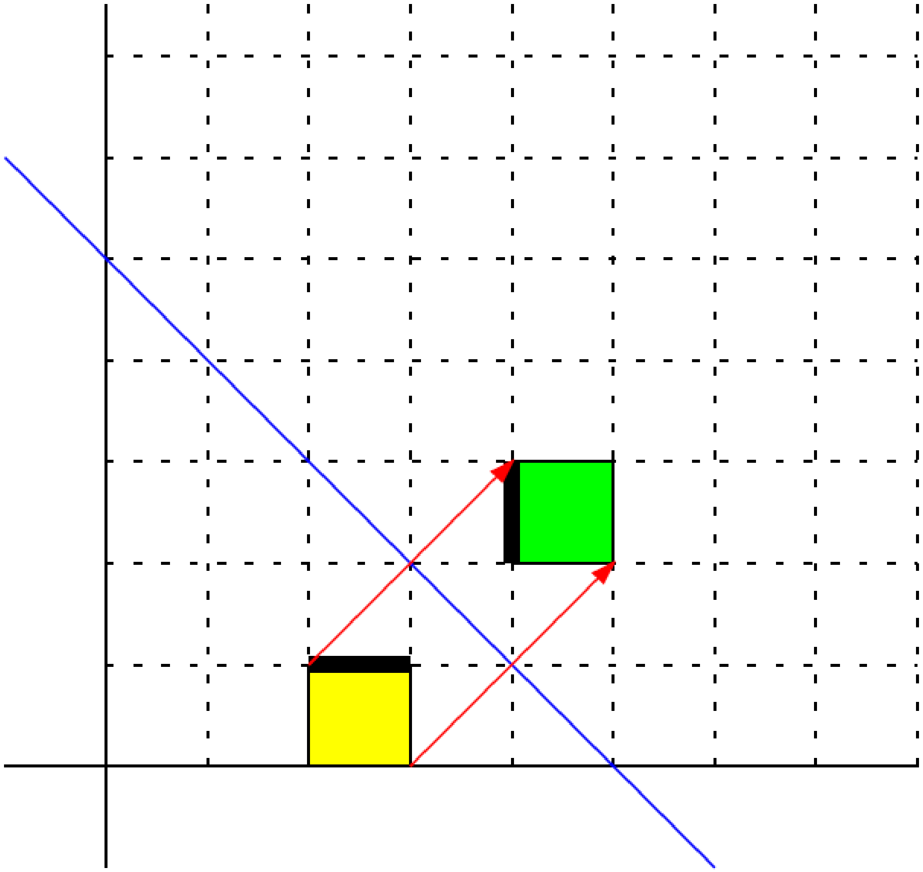
Homogeneous coordinates

- Translations in 2D can be represented by a shearing in 3D, by looking at the plane $z = 1$
- The matrix for a translation over the vector $t = \begin{bmatrix} x_t \\ y_t \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix}$
- Q: How should we represent points? And vectors?

Affine transformations

- Q: What is the matrix for the reflection in the line $y = -x + 5$?

- Hint: move the line to the origin, reflect and move the line back



Affine transformations

- Solution

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 5 \\ -1 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

- The rightmost matrix of the three translates over $(-5 \;\; 0)^T$, the leftmost matrix translates back over $(5 \;\; 0)^T$

Affine transformations

- The matrix for reflection in the line $y = -x + 5$ is

$$\begin{bmatrix} 0 & -1 & 5 \\ -1 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

- Q: But what if we translate by $(-4 \quad -1)^T$? This also makes the line $y = -x + 5$ go through the origin...

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Affine transformations

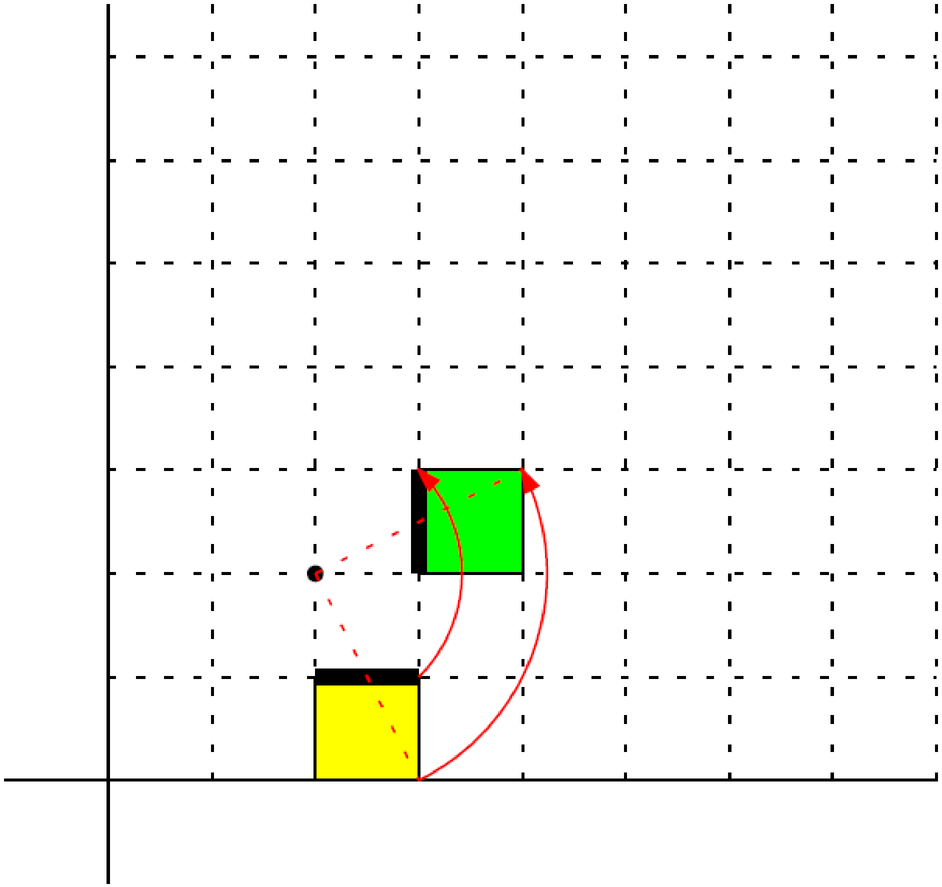
- The matrix for reflection in the line $y = -x + 5$ is

$$\begin{bmatrix} 0 & -1 & 5 \\ -1 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

- Q: What is the significance of the columns of the matrix?
- Does that give us a faster way to find matrices for affine transformations?

Affine transformations

- Q: What is the matrix for rotation about the point $(2, 2)$?



Transformations in 3D

- Transformations in 3D are very similar to those in 2D
  - For scaling, we have three scaling factors on the diagonal of the matrix
  - Reflection is done with respect to planes
  - Shearing can be done in either x-, y-, or z-direction (or a combination thereof)
  - Rotation is done about directed lines
  - For translations (and affine transformations in general), we use 4×4 matrices

Affine transformations in 3D

- A matrix for affine transformations in 3D looks like

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & t_1 \\ a_{21} & a_{22} & a_{23} & t_2 \\ a_{31} & a_{32} & a_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is the linear part and $\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$ is where the origin ends up due to the affine transformation

Extra terminology

- Some other terms that are important in linear algebra
  - Linear subspace: lower-dimensional linear space that includes the origin (or the whole space)
  - Kernel and image of a linear transformation: what maps to the origin, and the linear subspace where all vectors are mapped to
  - Rank of a matrix: number of linearly independent columns
  - Eigenvalue λ and eigenvector v such that $Av = \lambda v$
- When you need to know more, look in any linear algebra textbook